



# MURANG'A UNIVERSITY OF TECHNOLOGY

## SCHOOL OF PURE AND APPLIED SCIENCE

DEPARTMENT OF APPLIED SCIENCE

UNIVERSITY ORDINARY EXAMINATION

2017/2018 ACADEMIC YEAR

**FOURTH** YEAR ~~SECOND~~ SEMESTER EXAMINATION FOR BACHELOR OF  
SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE

SMA2407: FUNCTIONAL ANALYSIS

DURATION: 2 HOURS

DATE: 25<sup>TH</sup> APRIL 2018

TIME: 8.00AM – 11.00AM.

### Instructions to Candidates:

1. Answer **Section A** and **Any Other Two** questions in **Section B**.
2. Mobile phones are not allowed in the examination room.
3. You are not allowed to write on this examination question paper.

**SECTION A – ANSWER ALL QUESTIONS IN THIS SECTION (30 Marks)**

**QUESTION ONE**

- a) Define the following
- i. Metric Space (2 Marks)
  - ii. Normed space and Banach space (2 Marks)
  - iii. Equivalent norms (2 Marks)
- b) Show that a subspace  $Y$  of a Banach Space  $X$  is complete if and only if  $Y$  is closed. (5 Marks)
- c) Given an operator  $T: X \rightarrow Y$  defined by  $Tx = 3x$ . Show that  $T$  is linear. (3 Marks)
- d) Show that;
- i.  $|||x|| - ||y|| \leq ||x - y||$  for all  $x, y, \& X$  (2 Marks)
  - ii.  $|||x|| - ||y||| \leq ||x + y||$  for all  $x, y, \& X$  (2 Marks)
- e) Show that every convergent sequence in a normed space is a Cauchy sequence. (5 Marks)
- f) If  $[P_n]$  is a sequence of paranorms on a linear space  $X$ , then

$$P(X) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{P_n(X)}{1 + P_n(X)}$$

is a paranorm on  $X$ . (7 Marks)

**SECTION B – ANSWER ANY TWO QUESTIONS IN THIS SECTION**

**QUESTION TWO (20 MARKS)**

- a) Show that a norm on  $X$  defines a metric on  $X$  given by  $d(x, y) = ||x - y||$  (6 Marks)
- b) Prove the equivalent norms theorem which states that on a finite dimensional vector space  $X$ , any norm  $||\cdot||$  is equivalent to any other norm  $||\cdot||$  (7 Marks)
- c) Prove that  $Tx = \frac{1}{x}$  is not uniformly continuous on interval  $(0,1)$  (7 Marks)

**QUESTION THREE (20 MARKS)**

- a) Show that  $f(x) = \int_a^b x(t)dt, x \in [a, b]$  is linear and bounded functional on  $C[a, b]$  the set of continuous real valued functions on  $[a, b]$  (7 Marks)
- b) Let  $X = C$  be the space of all convergent sequences, show that  $||x||$  is a norm on  $C$  where  $||x|| = (\sum_{n=1}^{\infty} |X_n|^p)^{1/p}$  is a norm. (7 Marks)
- c) Let  $T: D(T) \rightarrow Y$  be a linear operator where  $D(T) \subset X$  and  $X, Y$  be normed spaces. Show that  $T$  is continuous if and only if  $T$  is bounded. (6 Marks)

**QUESTION FOUR (20 MARKS)**

- a) Define an inner product space. (3 Marks)
- b) Show that the space  $C^n$  (unitary space) with the function  $\langle, \rangle$  defined for arbitrary vectors  $W = (w_1, w_2, \dots, w_n)$   $Z = z_1, z_2, \dots, z_n$  in  $C^n$  by  $\langle w, z \rangle = \sum_{i=1}^n w_i \bar{z}_i$  is an inner product. (5 Marks)
- c) Show that an inner product on a space  $E$  defines a norm on  $E$  given by  $\|x\| = \sqrt{\langle x, x \rangle}$  (7 Marks)
- d) If in an inner product space  $x_n \rightarrow x$  and  $y_n \rightarrow y$  then show that  $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$  (5 Marks)

**QUESTION FIVE (20 MARKS)**

- a) Show that a compact subset  $M$  of a metric space is closed and bounded. (6 Marks)
- b) If a normed space  $X$  has the property that the closed unit ball  $M = \{x \mid \|x\| \leq 1\}$  is compact, show that  $X$  is finite dimensional. (7 Marks)
- c) Let  $X, Y$  be vector spaces, both real or both complex. Let  $T: D(T) \rightarrow Y$  be a linear operator with domain  $D(T) \subset X$  and range  $R(T) \subset Y$ . Then show that;
- The inverse  $T^{-1}: R(T) \rightarrow D(T)$  exists if and only if  $Tx = 0 \rightarrow x = 0$  (4 Marks)
  - If  $T^{-1}$  exists it is a linear operator. (3 Marks)