MURANG'A UNIVERSITY OF TECHNOLOGY

SCHOOL OF PURE AND APPLIED SCIENCE

DEPARTMENT OF APPLIED SCIENCE

UNIVERSITY ORDINARY EXAMINATION

2017/2018 ACADEMIC YEAR

FOURTH YEAR **SECOND** SEMESTER EXAMINATION FOR BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE

SMA2407: FUNCTIONAL ANALYSIS

DURATION: 2 HOURS

DATE: 25TH APRIL 2018

TIME: 8.00AM – 11.00AM.

Instructions to Candidates:

1. Answer **Section A** and **Any Other Two** questions in **Section B**.
2. Mobile phones are not allowed in the examination room.
3. You are not allowed to write on this examination question paper.

SECTION A – ANSWER ALL QUESTIONS IN THIS SECTION (30 Marks)

QUESTION ONE

a) Define the following
   i. Metric Space (2 Marks)
   ii. Normed space and Banoch space (2 Marks)
   iii. Equivalent norms (2 Marks)

b) Show that a subspace Y of a Banoch Space X is complete if and only if Y is closed. (5 Marks)

c) Given an operator $T: X \to Y$ defined by $Tx = 3x$. Show that T is linear. (3 Marks)

d) Show that;
   i. $|\|x\|| - \|y\| \leq \|x - y\|$ for all $x, y, \& X$ (2 Marks)
   ii. $|\|x\| - \|y\|| \leq \|x + y\|$ for all $x, y, \& X$ (2 Marks)

e) Show that every convergent sequence in a normed space is a Cauchy sequence. (5 Marks)

f) If $[P_n]$ is a sequence of paranorms on a linear space X, then

$$P(X) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{P_n(X)}{1 + P_n(X)}$$

is a paranorm on X. (7 Marks)

SECTION B – ANSWER ANY TWO QUESTIONS IN THIS SECTION

QUESTION TWO (20 MARKS)

a) Show that a norm on X defines a metric on X given by $d(x, y) = \|x - y\|$ (6 Marks)

b) Proof the equivalent norms theorem which states that on a finite dimensional vector space X, any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|$ (7 Marks)

c) Proof that $Tx = \frac{1}{x}$ is not uniformly continuous on interval $(0,1)$ (7 Marks)

QUESTION THREE (20 MARKS)

a) Show that $f(x) = \int_a^b x(t)dt, x \in [a, b]$ is linear and bounded functional on $C[a, b]$ the set of continuous real valued functions on $[a, b]$ (7 Marks)

b) Let $X = C$ be the space of all convergent sequences, show that $\|x\|$ is a norm on C where $\|x\| = (\sum_{n=1}^{\infty} |X_n|^p)p$ is a norm. (7 Marks)

c) Let $T: D(T) \to Y$ be a linear operator where $D(T) C X$ and X, Y be normed spaces. Show that T is continuous if and only if T is bounded. (6 Marks)

QUESTION FOUR (20 MARKS)

a) Define an inner product space. (3 Marks)

b) Show that the space C^n (unitary space) with the function $<,>$ defined for arbitrary vectors $W = (w_1, w_2, \cdots w_n)\ Z = z_1, z_2, \cdots z_n$ in C^n by $< w, z >= \sum_{i=1}^{n} w_i \bar{z}_i$ is an inner product. (5 Marks)

c) Show that an inner product on a space E defines a norm on E given by $\|x\| = \sqrt{< x, x >}$ (7 Marks)

d) If in an inner product space $x_n \to x\ and\ y_n \to y\ then\ show\ that\ < x_n, y_n > \to < x, y >$ (5 Marks)

QUESTION FIVE (20 MARKS)

a) Show that a compact subject M of a metric space is closed and bounded. (6 Marks)

b) If a normed space X has the property that the closed unit ball $M = \{x | \|x\|| \leq y\}$ is compact, show that X is finite dimensional. (7 Marks)

c) Let X, Y be vector spaces, both real or both complex. Let $T: D(T) \to Y$ be a linear operator with domain $D(T)CX$ and range $R(T)CY$. Then show that;

i. The inverse $T^{-1}: R(f) \to D(T)$exists if and only if $Tx = 0 \to x = 0$ (4 Marks)

ii. If T^{-1} exists it is a linear operator. (3 Marks)