MURANG'A UNIVERSITY OF TECHNOLOGY

SCHOOL OF PURE AND APPLIED SCIENCE

DEPARTMENT OF APPLIED SCIENCE

UNIVERSITY ORDINARY EXAMINATION

2017/2018 ACADEMIC YEAR

FOURTH YEAR **FIRST** SEMESTER EXAMINATION FOR BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE

SMA2404 – FIELD THEORY

DURATION: 2 HOURS

DATE: 6TH DECEMBER 2017

TIME: 9.00AM – 11.00AM

Instructions to Candidates:

1. Answer **Question 1** and **Any Other Two** questions.
2. Mobile phones are not allowed in the examination room.
3. You are not allowed to write on this examination question paper.

SECTION A (Compulsory)

QUESTION ONE (30 Marks)

a) Differentiate between an integral domain and a field and hence give an example of an integral domain which is not a field. (5 Marks)

b) List all the units in $\mathbb{Z}_{12}$ (ring of integers module 12). (2 Marks)

c) Let F be a field, define what is meant by the characteristics of a field and hence give any two (2) fields of characteristic zero. (4 Marks)

d) Let E be a field extension of F, define what is meant by an element $\propto \in E$ is algebraic over F and hence show that $i \in \mathbb{C}$ is algebraic over $\mathbb{R}$. (4 Marks)

e) Find a polynomial $f(x) \in Q[x]$ such that $\propto = \sqrt{1+\sqrt{3}}$ is a root of $f(x)$. (3 Marks)

f) If $f(x)$ is a polynomial of degree greater than 1 and if $f(\propto) = 0$ for some $\propto \in F$, show that $f(x)$is reducible over F. (3 Marks)

g) State the Eisenstein irreducibility criterion and hence use it to show that $25x^5 - 9x^4 + 3x^2 - 12$ is irreducible over $\mathbb{Q}$. (5 Marks)

h) By solving for the irreducible monic polynomial $f(x) \in \mathbb{Q}(x)$ such that $\propto$ is a root of $f(x)$ find the degree of $\propto = \sqrt{\sqrt{5}-2}$ over $\mathbb{Q}$. (4 Marks)

SECTION B (Answer any two questions)

QUESTION TWO (20 Marks)

a) Show that any finite extension E over F is an algebraic extension of F. (4 Marks)

b) Let $\mathbb{Q}$ be the field of rationals;

   i. Explain in details the meaning of $\mathbb{Q}(\sqrt{3})$. (2 Marks)

   ii. Give a basis of $\mathbb{Q}(\sqrt{3})$over $\mathbb{Q}$. (1 Mark)

c) State what is meant by the degree of an algebraic element b of an extension E of a field F hence find the degree of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$. (6 Marks)

d) Find the degree and basis for $(\mathbb{Q}\sqrt{2}\ \sqrt[3]{2})$ over $\mathbb{Q}$. (5 Marks)

e) Define an algebraic closed field. (2 Marks)

QUESTION THREE (20 Marks)

a) i) Let E be an algebraic extension of a field F and let $\propto, \beta \in E$. Explain what is meant by the elements $\propto$ and β being conjugates over the field F. (2 Marks)

ii) Find all the conjugates of $\sqrt{1+\sqrt{2}}$ over $\mathbb{Q}$. (6 Marks)

b) Define what is meant by an irreducible polynomial $F(x) \in F[x]$ (2 Marks)

c) By considering an irreducible polynomial $f(x)$over $\mathbb{Z}_2$ of degree 3, construct $Gf(8)$. (6 Marks)

d) Define a simple extension K of a subfield F if E is a field extension of F and $\propto \in E$ and hence give an example of a simple field extension of the field of real numbers $\mathbb{R}$. (4 Marks)

QUESTION FOUR (20 Marks)

a) State what is meant by zero divisors and hence not all the zero divisors in $\mathbb{Z}_{12}$. (4 Marks)

b) Define the degree of a field extension K of a field F. (2 Marks)

c) i) State what is meant by the splitting field of a polynomial $f(x)$over a field F. (2 Marks)

ii) Find the splitting field of $X^4 - 5X^2 + 6$. (4 Marks)

d) Show that a field F is algebraically closed if and only if every non-zero polynomial in $f(x)$ factors into linear factors. (6 Marks)

e) Find the characteristics of $\mathbb{Z}_7$. (2 Marks)

QUESTION FIVE (20 Marks)

a) Show that if E is a finite extension of a field F and K ia a finite extension of E, then K is a finite extension of F and $[K:F] = [K:E][E:F]$. (10 Marks)

b) Let K be an extension of F and $b \in K$. Let b be a root of a polynomial $q(x) \in F(x)$, show that $(x-b)$divides $q(x)$. (3 Marks)

c) Show that the field $F = \mathbb{Q}(i, -i, \sqrt{5}, -\sqrt{5}$ is a simple extension given by $F = \mathbb{Q}(i + \sqrt{5})$. (7 Marks)